

RECURSION FORMULAS FOR HOMFLY AND KAUFFMAN INVARIANTS

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ABSTRACT. In this note we describe the recursion relations between two parameter HOMFLY and Kauffman polynomials of framed links. These relations correspond to embeddings of quantized universal enveloping algebras. The relation corresponding to embeddings $g_n \supset g_k \times sl_{n-k}$ where g_n is either so_{2n+1} , so_{2n} or sp_{2n} is new.

INTRODUCTION

The relation between knot polynomials and solutions to the Yang-Baxter equation corresponding to quantized universal enveloping algebras was first established in [3] for HOMFLY and in [8] for Kauffman polynomials. In these references HOMFLY and Kauffman polynomials for special values of the second parameter were written as a local state sum on a diagram of a link. This result was extended in [6] and [7] into the construction of polynomial invariants of tangles graphs based on the representation theory of quantized universal enveloping algebras.

Embeddings of simple Lie algebras which embed the Dynkin diagram of a Lie algebra into the Dynkin diagram of the other algebra induce embeddings of corresponding quantized universal enveloping algebras. For classical Lie algebras we have:

$$(0.1) \quad U_q(sl_{n-k}) \otimes U_q(sl_k) \subset U_q(sl_n)$$

$$(0.2) \quad U_q(sl_k) \otimes U_q(so_{2n-2k}) \subset U_q(so_{2n}), \quad U_q(sl_k) \otimes U_q(sp_{2n-2k}) \subset U_q(sp_{2n})$$

$$(0.3) \quad U_q(sl_k) \otimes U_q(so_{2n+1-2k}) \subset U_q(so_{2n+1})$$

for $k = 1, \dots, n-1$, and

$$(0.4) \quad U_q(sl_n) \subset U_q(so_{2n}), \quad U_q(sl_n) \subset U_q(sp_{2n}), \quad U_q(sl_n) \subset U_q(so_{2n+1})$$

For a classical Lie algebra the restriction of the defining fundamental representation (vector representations) of quantized universal enveloping algebras to a diagrammatically embedded subalgebra gives certain identities between HOMFLY and Kauffman polynomials when the second variable in these polynomials is specialized to the appropriate power of q . These relations were described in [6] for embeddings (0.1) and (0.4) and also for $sl_2 \subset g_2$.

These relations were extended to non-specialized HOMFLY polynomial (when the second variable is not specialized to a power of q) by F. Jaeger [2]. They were generalized to invariants of graphs colored by special representations by Kauffman and Vogel [5] and by Hao Wu [9] and [10].

The goal of this note is to give a complete list of such identities for two variable HOMFLY and Kauffman polynomials. One should expect that each of the functors constructed in this note can be categorified, see [9] and [10] for some results in this direction.

In the first section we construct the functor $\phi_{tq;t,q}$ from the category of HOMFLY skein modules $\mathcal{H}_{q;tq}$ to the category $\mathcal{H}_{q;t,q}$ which can be regarded as a shuffle tensor product of additive categories $\mathcal{H}_{q;t}$ and $\mathcal{H}_{q;q}$. This functor gives recursive relation for HOMFLY polynomials of framed links expressing $H_{q,tq}(\mathcal{L})$ in terms of a linear combination of $H_{q,t}(\mathcal{L}')$ for links \mathcal{L}' which are obtained from \mathcal{L} by a simple combinatorial procedure. This recursion formula corresponds to (0.1) with $k = 1$. In the second section we give the recursion formula corresponding to (0.1) for all k for HOMFLY polynomials as a functor from one skein category to another. Similarly, the recursion relations from section 3 correspond to embeddings (0.4). All this is an overview of [2] and of [6]. In section 4 we give the recursion formula for Kauffman polynomials corresponding to the embeddings (0.2) and (0.3) which is the main result of this paper.

1. RECURSION CORRESPONDING TO $sl_{n+1} \supset sl_n$

1.0.1. *The functor $\phi_{q,t,q}$.* The definition of HOMFLY polynomials via skein relations naturally extends to invariants of tangles with values in skein modules [8]. Let us recall this construction.

Choose a line $L \subset \mathbf{R}^2$. A tangle \mathcal{T} in $I \times \mathbf{R}^2$ with end points in $L \subset (\{0\} \times \mathbf{R}^2) \cup (\{1\} \times \mathbf{R}^2)$ is an equivalence class of an embedding $(S^1)^l \times I^k \subset I \times \mathbf{R}^2$ such that end points belong to $(\{0\} \times \mathbf{R}^2) \cup (\{1\} \times \mathbf{R}^2)$. Such embedding before taking the equivalence class are called *geometric tangles*. The equivalence is taken with respect to homeomorphisms trivial at the boundary. A framing can be thought as a homeomorphism class of a continuous section of the normal bundle to \mathcal{T} . The framing is called blackboard if the corresponding framed tangle has a representative where the framing at the endpoints of \mathcal{T} lies in the intersection of L^\perp and $(0, 0, 1)^\perp$.

A *diagram* of a framed tangle is the projection of a geometric tangle to $L \times [0, 1]$, assuming that the tangle has a blackboard framing.

Objects of the category of tangles \underline{Tan} are sequences $(\epsilon_1, \dots, \epsilon_n)$ with $\epsilon = \pm 1$.

Morphisms between $\{\epsilon\}$ and $\{\sigma\}$ are oriented framed tangles with the orientation which agrees with $\{\epsilon\}$ and $\{\sigma\}$ at the end points and such that the framing at the end points is orthogonal to $L \subset \mathbf{R}^2$ and and point to positive (with respect to the standard orientation of L direction). Such framing is called blackboard framing.

Note that morphisms in this category can be naturally identified with framed Reidemeister classes of diagrams of tangles.

The composition of morphisms is the gluing and then taking the homeomorphism class of the result of the gluing. The detail can be found in [8].

For a ring A , define the additive category \underline{Tan}_A as the category with objects being direct sums of objects of \underline{Tan} but with morphisms being A -linear combination of tangles.

HOMFLY invariants are morphisms in quotient category of the $\underline{Tan}_{\mathbf{C}[t^{\pm 1}, q^{\pm 1}]}$ subject to the following relations:

$$(1.1) \quad \begin{array}{c} \text{Diagram 1} \end{array} - \begin{array}{c} \text{Diagram 2} \end{array} = (q - q^{-1}) \begin{array}{c} \text{Diagram 3} \end{array}$$

and

$$(1.2) \quad \begin{array}{c} \text{Diagram 4} \end{array} = t \begin{array}{c} \text{Diagram 5} \end{array}, \quad \begin{array}{c} \text{Diagram 6} \end{array} = t^{-1} \begin{array}{c} \text{Diagram 7} \end{array}$$

Objects of the resulting quotient category $\mathcal{H}_{q,t}$ are the same as objects in the category of oriented framed tangles, i.e. sequences $(\epsilon_1, \dots, \epsilon_n)$ where $\epsilon_i = \pm 1$. Morphisms in this category are skein classes of linear combinations of tangles. To indicate that the parameter involved in twist relations is t we will write t as a color of each connected component of of a tangle and will write $((\epsilon_1, t), \dots, (\epsilon_n, t))$ for objects.

Remark 1.1. The morphism between two identity objects (between two empty sequences) given by an unknot with trivial framing is determined by the skein relations and is equal to $(t - t^{-1})/(q - q^{-1})$.

Remark 1.2. When $t = q^n$ the category $\mathcal{H}_{q,t}$ has the quotient category which is naturally equivalent to the category of modules over $U_q(sl_n)$.

Define the category $\mathcal{H}_{q;s_1, \dots, s_k}$ as an additive braided monoidal category over $\mathbf{C}[q^{\pm 1}, s_1^{\pm 1}, \dots, s_k^{\pm 1}]$ with objects $((\epsilon_1, s_{i_1}), \dots, (\epsilon_n, s_{i_n}))$. Morphisms between two such objects A and B are quotients of linear combination of tangles with blackboard framing whose orientation agrees with the signs of A and B , whose colors agree with colors $\{s_{i_a}\}$ of A and B . The quotient is taken with respect to the HOMFLY relations which are (1.1) when both components are colored by the same variable s_i ; when $i \neq j$, we impose

$$(1.3) \quad \begin{array}{c} \swarrow \quad \searrow \\ s_i \quad s_j \\ \swarrow \quad \searrow \end{array} = \begin{array}{c} \swarrow \quad \searrow \\ s_i \quad s_j \\ \swarrow \quad \searrow \end{array} = \begin{array}{c} \swarrow \quad \searrow \\ s_i \quad s_j \\ \swarrow \quad \searrow \end{array}$$

Define the mapping $\phi_{q;t,q} : \mathcal{H}_{q;t,q} \rightarrow \mathcal{H}_{q;t,q}$ which acts on objects as $\phi_{q;t,q}((\epsilon_1, t), \dots, (\epsilon_n, t)) = \oplus((\epsilon_1, u_1), \dots, (\epsilon_n, u_n))$. Here the sum is taken over all values of u_i which are either q or t and the summation is over all such possibilities. On elementary morphisms we define $\phi_{q;t,q}$ as follows.

$$(1.4) \quad \phi_{q;t,q} \left(\begin{array}{c} \swarrow \quad \searrow \\ tq \quad tq \\ \swarrow \quad \searrow \end{array} \right) = \begin{pmatrix} \begin{array}{c} \swarrow \quad \searrow \\ t \quad t \\ \swarrow \quad \searrow \end{array} & 0 & 0 & 0 \\ 0 & (q - q^{-1}) \begin{array}{c} \swarrow \quad \searrow \\ t \quad q \\ \swarrow \quad \searrow \end{array} & \begin{array}{c} \swarrow \quad \searrow \\ t \quad q \\ \swarrow \quad \searrow \end{array} & 0 \\ 0 & \begin{array}{c} \swarrow \quad \searrow \\ q \quad t \\ \swarrow \quad \searrow \end{array} & 0 & 0 \\ 0 & 0 & 0 & \begin{array}{c} \swarrow \quad \searrow \\ q \quad q \\ \swarrow \quad \searrow \end{array} \end{pmatrix}$$

$$(1.5) \quad \phi_{q;t,q} \left(\begin{array}{c} \swarrow \quad \searrow \\ tq \quad tq \\ \swarrow \quad \searrow \end{array} \right) = \begin{pmatrix} \begin{array}{c} \swarrow \quad \searrow \\ t \quad t \\ \swarrow \quad \searrow \end{array} & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{c} \swarrow \quad \searrow \\ t \quad q \\ \swarrow \quad \searrow \end{array} & 0 \\ 0 & \begin{array}{c} \swarrow \quad \searrow \\ q \quad t \\ \swarrow \quad \searrow \end{array} & -(q - q^{-1}) \begin{array}{c} \swarrow \quad \searrow \\ q \quad t \\ \swarrow \quad \searrow \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \swarrow \quad \searrow \\ q \quad q \\ \swarrow \quad \searrow \end{array} \end{pmatrix}$$

$$(1.6) \quad \phi_{q;t,q} \left(\begin{array}{c} \downarrow \\ tq \\ \downarrow \end{array} \right) = \begin{pmatrix} \begin{array}{c} \downarrow \\ t \\ \downarrow \end{array} & 0 \\ 0 & \begin{array}{c} \downarrow \\ q \\ \downarrow \end{array} \end{pmatrix}$$

$$(1.7) \quad \phi_{q;t,q} \left(\begin{array}{c} \nearrow \quad \nwarrow \\ tq \end{array} \right) = \begin{pmatrix} \begin{array}{c} \nearrow \quad \nwarrow \\ t \end{array} \\ 0 \\ 0 \\ \begin{array}{c} \nearrow \quad \nwarrow \\ q \end{array} \end{pmatrix}, \quad \phi_{q;t,q} \left(\begin{array}{c} \nwarrow \quad \nearrow \\ tq \end{array} \right) = \begin{pmatrix} q \begin{array}{c} \nwarrow \quad \nearrow \\ t \end{array} \\ 0 \\ 0 \\ t^{-1} \begin{array}{c} \nwarrow \quad \nearrow \\ q \end{array} \end{pmatrix}$$

$$(1.8) \quad \phi_{q;t,q} \left(\begin{array}{c} \nearrow \quad \nwarrow \\ tq \end{array} \right) = \left(\begin{array}{c} \nearrow \quad \nwarrow \\ t \end{array} \quad 0 \quad 0 \quad \begin{array}{c} \nearrow \quad \nwarrow \\ q \end{array} \right), \quad \phi_{q;t,q} \left(\begin{array}{c} \nwarrow \quad \nearrow \\ tq \end{array} \right) = \left(q^{-1} \begin{array}{c} \nwarrow \quad \nearrow \\ t \end{array} \quad 0 \quad 0 \quad t \begin{array}{c} \nwarrow \quad \nearrow \\ q \end{array} \right)$$

$$(1.9) \quad \phi_{q;t,q} \left(\begin{array}{c} \text{tq} \quad \text{tq} \\ \downarrow \quad \downarrow \end{array} \right) = \begin{pmatrix} \begin{array}{c} \text{t} \quad \text{t} \\ \downarrow \quad \downarrow \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \text{t} \quad \text{q} \\ \downarrow \quad \downarrow \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} \text{q} \quad \text{t} \\ \downarrow \quad \downarrow \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \text{q} \quad \text{q} \\ \downarrow \quad \downarrow \end{array} \end{pmatrix}$$

Numerical factors are always to the left of the diagram and in a bigger script.

The matrices on the right side indicate the morphisms between various summands in the image of objects.

Theorem 1.1. *The mapping $\phi_{q;t,q}$ extends uniquely to a covariant functor of braided monoidal categories $\mathcal{H}_{q;tq} \rightarrow \mathcal{H}_{q;t,q}$.*

Proof. Composing elementary diagrams we extend the mapping $\phi_{q;t,q}$ to all objects. It is clear that if it is consistent with HOMFLY relations, then such extension exists and unique. For the positive twist

relation the consistency is a simple computation:

$$\begin{aligned}
 (1.10) \quad \phi_{q;t,q} \left(\begin{array}{c} \text{diagram: a crossing with a loop on the top-right strand labeled } tq \end{array} \right) &= \phi_{q;t,q} \left(\begin{array}{c} \text{diagram: a crossing with loops on both strands labeled } tq \end{array} \right) \\
 (1.11) \quad &= \begin{pmatrix} \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } tq \end{array} & \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } q \end{array} & 0 \\ q^{-1} & + t & (q - q^{-1}) \\ 0 & & \begin{array}{c} \text{diagram: crossing with loops } q \text{ and } tq \end{array} \end{pmatrix} \\
 (1.12) \quad &= \begin{pmatrix} \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } tq \end{array} & \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } q \end{array} & 0 \\ q^{-1} & + t(q - q^{-1}) & \\ 0 & & \begin{array}{c} \text{diagram: crossing with loops } q \text{ and } tq \end{array} \end{pmatrix} \\
 (1.13) \quad &= \begin{pmatrix} \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } tq \end{array} & \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } q \end{array} & 0 \\ q^{-1}t & + t(q - q^{-1}) & \\ 0 & & tq \end{pmatrix} \\
 (1.14) \quad &= \begin{pmatrix} \begin{array}{c} \text{diagram: crossing with loops } t \text{ and } tq \end{array} & 0 \\ tq & \\ 0 & \begin{array}{c} \text{diagram: crossing with loops } q \text{ and } tq \end{array} \end{pmatrix} = tq \cdot \phi_{q;t,q} \left(\begin{array}{c} \text{diagram: a crossing} \end{array} \right)
 \end{aligned}$$

Here we used the normalization $\phi_{q;t,q} \left(\begin{array}{c} \text{diagram: a circle with label } q \end{array} \right) = 1$, which agrees with skein relations.

The consistency of the negative twist relations with the functor $\phi_{q;t,q}$ is an almost identical calculation. The consistency of the skein relation with the functor $\phi_{q;t,q}$ is given below:

$$\phi_{q;t,q} \left(\begin{array}{c} \text{diagram: crossing with loops } tq \text{ and } tq \end{array} \right) - \phi_{q;t,q} \left(\begin{array}{c} \text{diagram: crossing with loops } tq \text{ and } tq \end{array} \right)$$

$$\begin{aligned}
&= \begin{pmatrix} \begin{array}{c} \text{diag}_1 \\ \text{diag}_2 \end{array} - \begin{array}{c} \text{diag}_3 \\ \text{diag}_4 \end{array} & 0 & 0 & 0 \\ 0 & (q - q^{-1}) \begin{array}{c} \text{diag}_5 \\ \text{diag}_6 \end{array} & 0 & 0 \\ 0 & 0 & (q - q^{-1}) \begin{array}{c} \text{diag}_7 \\ \text{diag}_8 \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \text{diag}_9 \\ \text{diag}_{10} \end{array} - \begin{array}{c} \text{diag}_{11} \\ \text{diag}_{12} \end{array} \end{pmatrix} \\
&= \begin{pmatrix} (q - q^{-1}) \begin{array}{c} \text{diag}_{13} \\ \text{diag}_{14} \end{array} & 0 & 0 & 0 \\ 0 & (q - q^{-1}) \begin{array}{c} \text{diag}_{15} \\ \text{diag}_{16} \end{array} & 0 & 0 \\ 0 & 0 & (q - q^{-1}) \begin{array}{c} \text{diag}_{17} \\ \text{diag}_{18} \end{array} & 0 \\ 0 & 0 & 0 & (q - q^{-1}) \begin{array}{c} \text{diag}_{19} \\ \text{diag}_{20} \end{array} \end{pmatrix} \\
&= (q - q^{-1}) \begin{pmatrix} \begin{array}{c} \text{diag}_{21} \\ \text{diag}_{22} \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \text{diag}_{23} \\ \text{diag}_{24} \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} \text{diag}_{25} \\ \text{diag}_{26} \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \text{diag}_{27} \\ \text{diag}_{28} \end{array} \end{pmatrix} = \phi_{q;t,q} \left(\begin{array}{c} \text{diag}_{29} \\ \text{diag}_{30} \end{array} \right)
\end{aligned}$$

□

1.0.2. *The recursion relation for invariants of links.* Let \mathcal{L} be an oriented framed link and $D_{\mathcal{L}}$ be its diagram with the blackboard framing. The image $[D_{\mathcal{L}}]_{q,t}$ of $D_{\mathcal{L}}$ in the skein module $\text{Hom}_{\mathcal{H}_{q;t}}(1, 1)$, where 1 is the identity object (empty sequence), is the HOMFLY invariant $H_{q,t}(\mathcal{L})$ of \mathcal{L} . That is

$$H_{q,t}(\mathcal{L}) = [D_{\mathcal{L}}]_{q,t}$$

Note that the corresponding invariant of oriented but unframed links is

$$\langle [\mathcal{L}] \rangle = t^{n_+ - n_-} H_{q,t}(\mathcal{L}),$$

where n_+ is the number of \nearrow extrema and n_- is the number of \searrow extrema.

The functor $\phi_{q;t,q}$ gives the recursion relation for HOMFLY polynomials of links. For the invariant of framed links we have:

$$(1.15) \quad H_{q,tq}(\mathcal{L}) = \sum_{D'} w(D_{\mathcal{L}}, D')_{q,t} [D']_{q,t}$$

Here the summation is taken over all diagrams D' which are obtained from $D_{\mathcal{L}}$ by all possible replacements of elementary diagrams colored by qt by elementary diagrams colored by t and q according to the way how functor $\phi_{q;t,q}$ acts and $w(D_{\mathcal{L}}, D')_{q,t}$ is an integer coefficients polynomial of variables q and t .

2. RECURSION CORRESPONDING TO $sl_n \supset sl_k \times sl_{n-k}$

In this section we will generalize the results of the previous section by constructing the functor $\phi_{q;s,u}$ between skein categories $\mathcal{H}_{q;su}$ and $\mathcal{H}_{q;s,u}$. The category $\mathcal{H}_{q;s,u}$ is defined in the previous section.

Object $(\varepsilon_1, \dots, \varepsilon_n)$ of $\mathcal{H}_{q;s,u}$ is mapped by $\phi_{q;s,u}$ to $\oplus((\epsilon_1, s_1), \dots, (\epsilon_n, s_n))$ where the sum is taken over all possible substitutions $s_i = s$ or $s_i = u$.

Let $t = su$. Define the functor on elementary diagrams as

$$(2.1) \quad \phi_{q;s,u} \left(\begin{array}{c} \text{diagram with } t \text{ strands} \end{array} \right) = \begin{pmatrix} \begin{array}{c} \text{diagram with } s \text{ strands} \\ \text{diagram with } s \text{ strands} \end{array} & 0 & 0 & 0 \\ 0 & (q - q^{-1}) \begin{array}{c} \text{diagram with } s \text{ and } u \text{ strands} \end{array} & \begin{array}{c} \text{diagram with } s \text{ and } u \text{ strands} \end{array} & 0 \\ 0 & \begin{array}{c} \text{diagram with } u \text{ and } s \text{ strands} \end{array} & 0 & 0 \\ 0 & 0 & 0 & \begin{array}{c} \text{diagram with } u \text{ strands} \end{array} \end{pmatrix}$$

$$(2.2) \quad \phi_{q;s,u} \left(\begin{array}{c} \text{diagram with } t \text{ strands} \end{array} \right) = \begin{pmatrix} \begin{array}{c} \text{diagram with } s \text{ strands} \\ \text{diagram with } s \text{ strands} \end{array} & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{c} \text{diagram with } s \text{ and } u \text{ strands} \end{array} & 0 \\ 0 & \begin{array}{c} \text{diagram with } u \text{ and } s \text{ strands} \end{array} & -(q - q^{-1}) \begin{array}{c} \text{diagram with } u \text{ and } s \text{ strands} \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \text{diagram with } u \text{ strands} \end{array} \end{pmatrix}$$

$$(2.3) \quad \phi_{q;s,u} \left(\begin{array}{c} \text{diagram with } t \text{ strands} \end{array} \right) = \begin{pmatrix} \begin{array}{c} \text{diagram with } s \text{ strands} \\ \text{diagram with } s \text{ strands} \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \text{diagram with } s \text{ and } u \text{ strands} \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} \text{diagram with } u \text{ and } s \text{ strands} \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \text{diagram with } u \text{ strands} \end{array} \end{pmatrix}$$

$$(2.4) \quad \phi_{q;s,u} \left(\begin{array}{c} \text{diagram with } t \text{ strands} \end{array} \right) = \begin{pmatrix} \begin{array}{c} \text{diagram with } s \text{ strands} \\ \text{diagram with } s \text{ strands} \end{array} & 0 \\ 0 & \begin{array}{c} \text{diagram with } u \text{ strands} \end{array} \end{pmatrix}$$

$$(2.5) \quad \phi_{q;s,u} \left(\begin{array}{c} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{array} \right) = \begin{pmatrix} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{pmatrix}, \quad \phi_{q;s,u} \left(\begin{array}{c} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{array} \right) = \begin{pmatrix} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{pmatrix}$$

$$(2.6) \quad \phi_{q;s,u} \left(\begin{array}{c} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{array} \right) = \begin{pmatrix} \text{cup} & \text{0} & \text{0} & \text{cup} \end{pmatrix}, \quad \phi_{q;s,u} \left(\begin{array}{c} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{array} \right) = \begin{pmatrix} \text{cup} & \text{0} & \text{0} & \text{cup} \end{pmatrix}$$

The following statement is a straightforward generalization of the corresponding theorem from the previous section.

Theorem 2.1. *There exists unique covariant braided monoidal functor $\mathcal{H}_{q;su} \rightarrow \mathcal{H}_{q;s,u}$ which acts on elementary diagrams as above.*

When $s = q^k$ and $u = q^{n-k}$ the functor $\phi_{q;s,u}$ restricts to the functor between quotient categories and becomes the restriction functor from the category of $U_q(sl_n)$ modules to the category of $U_q(sl_k) \otimes U_q(sl_{n-k})$ modules.

3. RECURSION CORRESPONDING TO $so_{2n} \supset sl_n$ AND $sp_{2n} \supset sl_n$

Let \underline{Tan}' be the category of non-oriented framed tangles. Objects in this category are integers, morphisms between objects (n) and (m) are non-oriented framed tangles with blackboard framing at the ends, with n upper ends and with m lower ends. For a ring A , define \underline{Tan}'_A as the additive A -linear category where objects are direct sum of objects in \underline{Tan}' and morphisms are linear combination of morphisms from \underline{Tan}' .

Kauffman invariants are morphisms in the quotient category $\mathcal{K}_{q;s}$ of $\underline{Tan}'_{\mathbb{C}[q^{\pm 1}, s^{\pm 1}]}$ subject to the following relations

$$(3.1) \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = (q - q^{-1}) \begin{pmatrix} \diagup \diagdown \\ \diagdown \diagup \end{pmatrix} \begin{pmatrix} \diagdown \diagup \\ \diagup \diagdown \end{pmatrix}$$

and

$$(3.2) \quad \begin{pmatrix} \text{loop} \end{pmatrix} = s, \quad \begin{pmatrix} \text{loop} \end{pmatrix} = s^{-1}$$

The following theorem describe the covariant functor $\psi_{q;t}$ from $\mathcal{K}_{q;t^2q^{-1}}$ to $\mathcal{H}_{q;t}$. The category $\mathcal{K}_{q;s}$ is a braided monoidal category. Its morphisms are compositions of tensor products of elementary morphisms [7]

Theorem 3.1. *There exists unique functor $\psi_{q;t}$ of braided monoidal categories $\mathcal{K}_{q;t^2q^{-1}} \rightarrow \mathcal{H}_{q;t}$ which acts on objects as $\psi_{q;t}(n) = \bigoplus_{\varepsilon_1, \dots, \varepsilon_n = \pm} (\varepsilon_1, \dots, \varepsilon_n)$ with its action on elementary morphisms described below*

$$(3.3) \quad \psi_{q;t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \begin{pmatrix} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{array} + \begin{array}{c} \text{cup} \\ \text{0} \\ \text{0} \\ \text{cup} \end{array} & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 \\ 0 & \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} & 0 & 0 \\ 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \end{pmatrix}$$

$$(3.4) \quad \psi_{q;t} \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = \begin{pmatrix} \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} & 0 \\ 0 & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} & \begin{array}{c} \curvearrowright \\ c \end{array} + \begin{array}{c} \curvearrowleft \\ d \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \end{pmatrix}$$

$$(3.5) \quad \psi_{q;t} \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \begin{pmatrix} \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \end{pmatrix}$$

$$(3.6) \quad \psi_{q;t} \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \begin{pmatrix} 0 \\ tq^{-1} \end{pmatrix}, \quad \psi_{q;t} \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \begin{pmatrix} 0 & t^{-1}q & \curvearrowright & \curvearrowleft & 0 \end{pmatrix}$$

Here coefficients a , b , c and d are

$$(3.7) \quad a = q - q^{-1}, b = -(q - q^{-1})t^{-1}q, c = -(q - q^{-1}) \text{ and } d = (q - q^{-1})tq^{-1}$$

Proof. To prove the theorem we should check skein relation. Here, as in section 1 we assume that the functor $\psi_{q;t}$ bring the left side of equalities of diagrams below to the right side. Now, let us check skein relations. The image of the left side of (3.1) in $\mathcal{H}_{q;t}$ is

$$\begin{pmatrix} \begin{array}{c} \diagdown \quad \diagup \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \diagup \quad \diagdown \\ \diagup \quad \diagdown \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \curvearrowright \\ a \end{array} + \begin{array}{c} \curvearrowleft \\ b \end{array} & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} & 0 \\ 0 & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} & -c \begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} - d \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} - \begin{array}{c} \diagdown \quad \diagup \\ \diagup \quad \diagdown \end{array} \end{pmatrix}$$

$$= \begin{pmatrix} (q - q^{-1}) \downarrow \downarrow & 0 & 0 & 0 \\ 0 & a \downarrow \uparrow + b \downarrow \downarrow & \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} - \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} & 0 \\ 0 & \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} - \begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} & -c \downarrow \uparrow - d \downarrow \downarrow & 0 \\ 0 & 0 & 0 & (q - q^{-1}) \uparrow \uparrow \end{pmatrix}$$

The functor $\psi_{q,t}$ maps the right hand side of (3.1) to

$$(q - q^{-1}) \begin{pmatrix} \downarrow \downarrow & 0 & 0 & 0 \\ 0 & \downarrow \uparrow - t^{-1}q \downarrow \downarrow & - \downarrow \downarrow & 0 \\ 0 & - \downarrow \downarrow & \downarrow \downarrow - tq^{-1} \downarrow \downarrow & 0 \\ 0 & 0 & 0 & \uparrow \uparrow \end{pmatrix}$$

Comparing the coefficients of these diagrams for values of a, b, c and d given in the theorem, we have the desired identity of $\psi_{q,t}$.

Framing moves can be checked similarly. For example, for one of the nontrivial components of the image of the left move in (3.2) we have:

$$\begin{aligned} & \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} t^{-1}q + (a \downarrow \uparrow + b \downarrow \downarrow) tq^{-1} \begin{array}{c} \uparrow \\ \downarrow \end{array} = \begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} t^{-1}q + a \downarrow \uparrow \begin{array}{c} \uparrow \\ \downarrow \end{array} \begin{array}{c} \uparrow \\ \downarrow \end{array} tq^{-1} + b \downarrow \downarrow \begin{array}{c} \uparrow \\ \downarrow \end{array} tq^{-1} \\ &= \begin{array}{c} \uparrow \\ \downarrow \end{array} q + \begin{array}{c} \uparrow \\ \downarrow \end{array} a \frac{t-t^{-1}}{q-q^{-1}} tq^{-1} + b \downarrow \downarrow tq^{-1} = \begin{array}{c} \uparrow \\ \downarrow \end{array} [q + (t - t^{-1})tq^{-1} - (q - q^{-1})t^{-1}qtq^{-1}] = t^2q^{-1} \begin{array}{c} \uparrow \\ \downarrow \end{array} \\ &= s \begin{array}{c} \uparrow \\ \downarrow \end{array} \\ &\text{and} \end{aligned}$$

$$\begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} tq^{-1} = t \begin{array}{c} \uparrow \\ \downarrow \end{array} tq^{-1} = s \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

It is easy to check that other relations hold as well, in particular we have

$$\psi_{q,t} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right) = \psi_{q,t} \left(\begin{array}{c} \uparrow \\ \downarrow \end{array} \right), \quad \psi_{q,t} \left(\begin{array}{c} \nearrow \searrow \\ \searrow \nearrow \end{array} \right) = \psi_{q,t} \left(\begin{array}{c} \nearrow \nearrow \\ \searrow \searrow \end{array} \right)$$

To extend the map $\psi_{q,t}$ from elementary diagrams to any tangle, one should check relations between elementary diagrams. This is completely parallel to [8]. For example, here is the proof of invertibility of the braiding

$$\begin{aligned}
 \psi_{q;t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) &= \begin{pmatrix} \begin{array}{ccc} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 \\ 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & \left(a \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} + b \begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} + c \begin{array}{c} \diagup \diagdown \\ \curvearrowright \curvearrowleft \end{array} + d \begin{array}{c} \diagup \diagdown \\ \curvearrowleft \curvearrowright \end{array} \right) \\ 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \\ 0 & 0 & 0 \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \end{array} \end{pmatrix} \\
 &= \begin{pmatrix} \begin{array}{ccc} \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} & 0 & 0 \\ 0 & \begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} & \left(a \begin{array}{c} \diagup \diagdown \\ \curvearrowright \curvearrowleft \end{array} + bt \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} + c \begin{array}{c} \diagup \diagdown \\ \curvearrowleft \curvearrowright \end{array} + dt^{-1} \begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} \right) \\ 0 & 0 & \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} \\ 0 & 0 & 0 \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} \end{array} \end{pmatrix} \\
 &= \begin{pmatrix} \begin{array}{ccc} \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} & 0 & 0 \\ 0 & \begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} & [a(q - q^{-1}) + (bt + dt^{-1})] \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} \\ 0 & 0 & \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} \\ 0 & 0 & 0 \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} \end{array} \end{pmatrix} \\
 &= \begin{pmatrix} \begin{array}{cccc} \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} & 0 & 0 & 0 \\ 0 & \begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} & 0 & 0 \\ 0 & 0 & \begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} & 0 \\ 0 & 0 & 0 & \begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} \end{array} \end{pmatrix} = \psi_{q;t} \left(\begin{array}{c} \curvearrowright \curvearrowleft \\ \diagup \diagdown \end{array} \right) \left(\begin{array}{c} \curvearrowleft \curvearrowright \\ \diagup \diagdown \end{array} \right)
 \end{aligned}$$

Other relations can be checked similarly. \square

When $s = \pm q^{2n-1}$ and $t = q^n$, the functor $\psi_{q;t}$ restricts to the functor between corresponding quotient categories and becomes the restriction functor from $U_q(so_{2n})$ or $U_q(sp_{2n})$ modules to $U_q(sl_n)$ modules.

4. THE RECURSION CORRESPONDING TO $so_{2n} \supset so_{2k} \times sl_{n-k}$, $so_{2n+1} \supset so_{2k+1} \times sl_{n-k}$ AND $sp_{2n} \supset sp_{2k} \times sl_{n-k}$

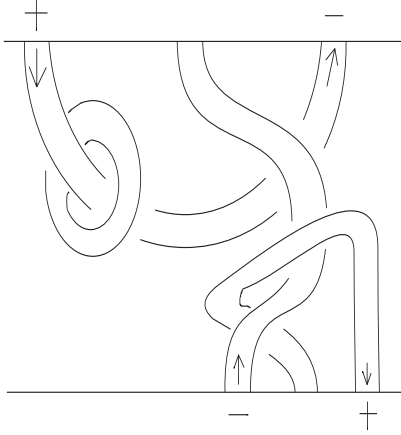
Let g_n be either so_{2n+1} , so_{2n} or sp_{2n} . In this section we will construct the functor between skein categories corresponding to embeddings $g_n \supset g_k \times sl_{n-k}$ of Lie algebras.

The skein category $\mathcal{K}_{q;s}$ corresponds to g_k and to Kauffman invariants we described in the previous section. Now we will define the skein category $\mathcal{HK}_{q;s,t}$ corresponding to products of HOMFLY and Kauffman invariants and will describe the functor

$$(4.1) \quad \chi_{q;s,t} : \mathcal{K}_{q;st^2} \rightarrow \mathcal{HK}_{q;s,t}$$

First, consider the category \underline{Tan}'' of framed tangles with some components being oriented and some not.

Objects of \underline{Tan}'' are sequences of $\{\varepsilon_1, \dots, \varepsilon_n\}$ where $\varepsilon_i = \pm, 0$. Morphisms between $\{\varepsilon\}$ and $\{\sigma\}$ are framed tangles with the blackboard framing near ends. Some components of these tangles are oriented, some not. The orientation of components agrees with the objects as it is shown on the following figure.



Objects of the category $\mathcal{HK}_{q;s,t}$ are direct sums of objects in \underline{Tan}'' . Morphisms between $\{\varepsilon\}$ and $\{\sigma\}$ are quotient spaces of linear combination of morphisms in \underline{Tan}' modulo defining relations as in $\mathcal{K}_{q;s}$ for non-oriented components, as in $\mathcal{H}_{q;t}$ for oriented components and with the following relations between oriented and non-oriented components

$$\begin{array}{c} \begin{array}{ccccccc} \diagdown & = & \diagdown & = & \diagdown & , & \diagdown \\ \diagup & & \diagup & & \diagup & & \diagup \end{array} \\ \begin{array}{ccccccc} \diagup & = & \diagup & = & \diagup & , & \diagup \\ \diagdown & & \diagdown & & \diagdown & & \diagdown \end{array} \end{array}$$

Now we will construct a covariant functor $\chi_{q;s,t} : \mathcal{K}_{q;st^2} \rightarrow \mathcal{HK}_{q;s,t}$. Define it on objects as $\chi_{q;s,t}(n) = \bigoplus_{\varepsilon_1, \dots, \varepsilon_n = \pm, 0} (\varepsilon_1, \dots, \varepsilon_n)$. To define the functor $\chi_{q;s,t}$ on morphisms it is convenient to introduce a formal variable a such that $s = a^2 q^{-1}$ and $p = q - q^{-1}$.

Then we define $\chi_{q;s,t}$ on elementary morphism as follows.

$$\chi_{q;s,t} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right)$$

$$= \begin{pmatrix} \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \begin{array}{c} 0 \\ \diagdown \diagup \end{array} & p \downarrow \uparrow \left(-ps^{-1}t^{-1} \right) \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & -pa^{-1} \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \\ 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & p \downarrow \uparrow \left(\begin{array}{c} \diagdown \diagup \\ 0 \end{array} \right) & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & p \downarrow \uparrow \left(\begin{array}{c} \diagdown \diagup \\ 0 \end{array} \right) & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 \\ 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 \\ 0 & -pt^{-1}a^{-1}q \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} \end{pmatrix}$$

$$\chi_{q;s,t} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right)$$

$$= \begin{pmatrix} \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & -p \downarrow \uparrow \left(\begin{array}{c} \diagup \diagdown \\ 0 \end{array} \right) + pst \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & ptaq^{-1} \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \\ 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 \\ 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & -p \downarrow \uparrow \left(\begin{array}{c} \diagup \diagdown \\ 0 \end{array} \right) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & -p \downarrow \uparrow \left(\begin{array}{c} \diagup \diagdown \\ 0 \end{array} \right) & 0 \\ 0 & 0 & pa \begin{array}{c} \diagup \diagdown \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} \end{pmatrix}$$

$$\chi_{q;s,t} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right) \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \right) = \begin{pmatrix} \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagdown \diagup \\ 0 \end{array} \end{pmatrix}$$

$$\chi_{q;s,t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \begin{pmatrix} 0 \\ t^{-1}a^{-1} \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \\ aq^{-1} \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ t^{-1} \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \end{pmatrix}, \chi_{q;s,t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \left(\begin{array}{c} 0 \\ a^{-1}q \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \\ ta \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ t \begin{array}{c} \diagup \diagdown \\ 0 \end{array} \end{array} \right)$$

Theorem 4.1. *The map $\chi_{q;s,t}$ extends uniquely to a covariant functor from $\mathcal{K}_{q;st^2}$ to $\mathcal{HK}_{q;s,t}$.*

Proof. To prove this we should check relations. Composing elementary diagrams we have:

$$\chi_{q;s,t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \begin{pmatrix} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & * & 0 & 0 & 0 & 0 & 0 & ** \\ 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} & 0 \\ 0 & 0 & *** & 0 & 0 & 0 & 0 & 0 & \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \end{pmatrix}$$

Matrix elements which are denotes by *, ** and *** are easy to compute:

$$\begin{aligned} * &= p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - ps^{-1}t^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + pst \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p^2 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \\ &= p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - ps^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + ps \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p^2 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} (1 + \frac{s-s^{-1}}{q-q^{-1}}) \\ &= p(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}) - ps^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + ps \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p^2 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} (1 + \frac{s-s^{-1}}{q-q^{-1}}) \\ &= p^2 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - ps^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + ps \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - p^2 \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} (1 + \frac{s-s^{-1}}{q-q^{-1}}) \\ &= 0 \\ ** &= ptaq^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - pa^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = paq^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - pa^{-1}s \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = 0 \\ *** &= -pt^{-1}a^{-1}q \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + pa \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = -pa^{-1}q \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} + pas^{-1} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = 0 \end{aligned}$$

Thus we proved

$$\chi_{q;s,t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \\ \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \chi_{q;s,t} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)$$

Framing moves can be checked similarly. For example, for one of the nontrivial components of the image of the left move in (3.2) we have:

$$\begin{aligned}
& \text{Diagram 1} \quad s^{-1}t^{-1} + (p) \quad \text{Diagram 2} \quad -ps^{-1}t^{-1} \quad \text{Diagram 3} \quad st + p \quad \text{Diagram 4} \quad tt^{-1} \\
& = \text{Diagram 5} \quad s^{-1}t^{-1} + (q - q^{-1}) \quad \text{Diagram 6} \quad st - (q - q^{-1}) + (q - q^{-1}) \\
& = \text{Diagram 7} \quad s^{-1} + \text{Diagram 8} \quad (q - q^{-1}) \frac{t-t^{-1}}{q-q^{-1}} st - (q - q^{-1}) + \text{Diagram 9} \quad (q - q^{-1}) [1 + \frac{s-s^{-1}}{q-q^{-1}}] \\
& = \text{Diagram 10} \quad (s^{-1} + st^2 - s - (q - q^{-1}) + (q - q^{-1}) + s - s^{-1})
\end{aligned}$$

$$\begin{aligned}
 &= st^2 \quad \text{(diagram: a single loop with an arrow pointing down)} \\
 &= p) \left(\text{diagram: a loop with an arrow pointing up} \right) st + \text{diagram: a crossing} \quad tt^{-1} \\
 &= (q - q^{-1}) \left(\text{diagram: a circle with an arrow pointing right} \right) st + \text{diagram: a loop with an arrow pointing up} \\
 &= (q - q^{-1}) \left(\frac{t-t^{-1}}{q-q^{-1}} st \right) + s \quad \text{(diagram: a loop with an arrow pointing up)} \\
 &= st^2 \quad \text{(diagram: a single loop with an arrow pointing down)}
 \end{aligned}$$

and

$$\text{diagram: a crossing} \quad st = st^2 \quad \text{(diagram: a single loop with an arrow pointing up)}$$

Remaining relations can be checked similarly. \square

When $s = \pm q^{N-1}$ and $t = q^{n-k}$, where $N = 2k + 1$ for so_{2k+1} and $N = 2k$ for so_{2k} and sp_{2k} , the category $\mathcal{K}_{q;st^2}$ has the quotient category which is naturally equivalent to the category of $U_q(g_n)$ -modules. Here $+$ is for so and $-$ is for sp . For these values of s and t the category $\mathcal{HK}_{q;s,t}$ is naturally equivalent to the category of $U_q(g_n)$ -modules regarded as $U_q(g_k) \otimes U_q(sl_{n-k})$ -modules and functor $\chi_{q;s,t}$ becomes the restriction functor.

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